

A REFINEMENT OF THE COMPANION OF OSTROWSKI INEQUALITY FOR FUNCTIONS OF BOUNDED VARIATION AND APPLICATIONS

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ABSTRACT. In this paper we establish a refinement of the companion of Ostrowski inequality for functions of bounded variation. Applications for the trapezoid inequality, the mid-point inequality, and to probability density functions are also given.

1. INTRODUCTION

In 1938, Ostrowski [21] established the following interesting integral inequality for differentiable mappings with bounded derivatives:

Theorem 1.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative is bounded on (a, b) and denote $\|f'\|_\infty = \sup_{t \in (a, b)} |f'(t)| < \infty$. Then, for all $x \in [a, b]$, one has the inequality*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty.$$

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

This inequality has attracted considerable interest over the years, and many authors proved generalizations, modifications and applications of it. In [10], Dragomir extended this result to the larger class of functions of bounded variation, as follows:

Theorem 1.2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a function of bounded variation on $[a, b]$. Denote by $\bigvee_a^b(f)$ its total variation on $[a, b]$. Then, for all $x \in [a, b]$, one has the inequality*

$$(1.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b(f).$$

The constant $\frac{1}{2}$ is sharp in the sense that it can not be replaced by a smaller one.

The best inequality one can obtain from (1.1) and (1.2) is the midpoint inequality. The corresponding version for the generalised trapezoid inequality was obtained by Cerone, Dragomir, and Pearce in [7], from which one can derive the trapezoid inequality. Recently, by using a critical Lemma, Dragomir [12] proved refinement of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation, the particular cases of which provide refinements of the trapezoid and mid-point inequalities.

In [14], Guessab and Schmeisser, in the effort of incorporating together the mid-point and trapezoid inequalities, proved a companion of Ostrowski's inequality. Motivated by [14], Dragomir [11] proved some companions of Ostrowski's inequality for functions of bounded variation, as follows:

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Theorem 1.3. Assume that the function $f : [a, b] \rightarrow \mathbb{R}$ is of bounded variation on $[a, b]$. Then the following inequalities

$$\begin{aligned}
 & \left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
 & \leq \frac{1}{b-a} \left[(x-a) \bigvee_a^x(f) + \left(\frac{a+b}{2} - x \right) \bigvee_x^{a+b-x}(f) + (x-a) \bigvee_{a+b-x}^b(f) \right] \\
 (1.3) \quad & \leq \left[\frac{1}{4} + \left| \frac{x - \frac{3a+b}{4}}{b-a} \right| \right] \bigvee_a^b(f)
 \end{aligned}$$

hold for all $x \in [a, \frac{a+b}{2}]$. The constant $\frac{1}{4}$ is best possible.

The best inequality one can obtain from (1.3) is the trapezoid type inequality, namely,

$$(1.4) \quad \left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{4} \bigvee_a^b(f).$$

Here the constant $\frac{1}{4}$ is also the best possible.

For other related results, the reader may refer to [1, 2, 3, 4, 5, 6, 13, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28] and the references therein.

The main aim of this paper is to establish a refinement of the companion of Ostrowski inequality (1.3) for functions of bounded variation. Applications for the trapezoid inequality, the mid-point inequality, the trapezoid type inequality (1.4), and to probability density functions are also given.

2. MAIN RESULTS

To prove our main results, we need the following lemma, which is a slight improvement of [12, Lemma 2.1].

Lemma 2.1. Let $u, f : [a, b] \rightarrow \mathbb{R}$. If u is continuous on $[a, b]$ and f is of bounded variation on $[c, b] \supseteq [a, b]$, then

$$\begin{aligned}
 (2.1) \quad & \left| \int_a^b u(t) df(t) \right| \leq \int_a^b |u(t)| d \left(\bigvee_c^t(f) \right) \\
 & \leq \left[\bigvee_a^b(f) \right]^{\frac{1}{q}} \left[\int_a^b |u(t)|^p d \left(\bigvee_c^t(f) \right) \right]^{\frac{1}{p}} \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\
 & \leq \max_{t \in [a, b]} |u(t)| \bigvee_a^b(f).
 \end{aligned}$$

Proof. See [12, Lemma 2.1]. □

The following result may be stated.

Theorem 2.1. Assume that the function $f : [a, b] \rightarrow \mathbb{R}$ is of bounded variation on $[a, b]$. Then

$$(2.2) \quad \left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq Q(x)$$

for all $x \in [a, \frac{a+b}{2}]$, where

$$\begin{aligned}
 Q(x) &:= \frac{1}{b-a} \left[2 \left(\frac{3a+b}{4} - x \right) \bigvee_x^{a+b-x} (f) + \int_a^{\frac{a+b}{2}} \operatorname{sgn}(x-t) \bigvee_t^{a+b-t} (f) dt \right] \\
 &\leq \frac{1}{b-a} \left[(x-a) \bigvee_a^x (f) + \left(\frac{a+b}{2} - x \right) \bigvee_x^{a+b-x} (f) + (x-a) \bigvee_{a+b-x}^b (f) \right] \\
 (2.3) \quad &\leq \left[\frac{1}{4} + \left| \frac{x - \frac{3a+b}{4}}{b-a} \right| \right] \bigvee_a^b (f).
 \end{aligned}$$

We also have

$$\begin{aligned}
 Q(x) &\leq \frac{1}{b-a} \left[\bigvee_a^b (f) \right]^{\frac{1}{q}} \left\{ \left[\left(\frac{a+b}{2} - x \right)^p - (x-a)^p \right] \bigvee_x^{a+b-x} (f) \right. \\
 &\quad \left. + p \int_a^{\frac{a+b}{2}} r_p(x, t) \operatorname{sgn}(x-t) \bigvee_t^{a+b-t} (f) dt \right\}^{\frac{1}{p}} \\
 &\leq \frac{1}{b-a} \left[\bigvee_a^b (f) \right]^{\frac{1}{q}} \left\{ (x-a)^p \bigvee_a^x (f) + \left(\frac{a+b}{2} - x \right)^p \bigvee_x^{a+b-x} (f) + (x-a)^p \bigvee_{a+b-x}^b (f) \right\}^{\frac{1}{p}} \\
 (2.4) \quad &\leq \left[\frac{1}{4} + \left| \frac{x - \frac{3a+b}{4}}{b-a} \right| \right] \bigvee_a^b (f),
 \end{aligned}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $r_p : [a, b]^2 \rightarrow \mathbb{R}$ with

$$(2.5) \quad r_p(x, t) := \begin{cases} (t-a)^{p-1}, & t \in [a, x], \\ \left(\frac{a+b}{2} - x \right)^{p-1}, & t \in \left(x, \frac{a+b}{2} \right]. \end{cases}$$

Proof. Define the kernel $K(x, t)$ by

$$(2.6) \quad K(x, t) := \begin{cases} t-a, & t \in [a, x], \\ t - \frac{a+b}{2}, & t \in (x, a+b-x], \\ t-b, & t \in (a+b-x, b], \end{cases}$$

for all $x \in [a, \frac{a+b}{2}]$. Using the integration by parts formula for Riemann-Stieltjes integrals, we obtain (see [11])

$$(2.7) \quad \frac{1}{b-a} \int_a^b K(x, t) df(t) = \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t) dt.$$

Now, if f is of bounded variation on $[a, b]$, then on taking the modulus, applying the first inequality in (2.1) and making a substitution of the form $t = a + b - s$, we deduce

$$\begin{aligned}
& \left| \int_a^b K(x, t) df(t) \right| \\
& \leq \left| \int_a^x (t - a) df(t) \right| + \left| \int_x^{a+b-x} \left(t - \frac{a+b}{2} \right) df(t) \right| + \left| \int_{a+b-x}^b (t - b) df(t) \right| \\
& \leq \int_a^x |t - a| d \left(\bigvee_a^t(f) \right) + \int_x^{a+b-x} \left| t - \frac{a+b}{2} \right| d \left(\bigvee_a^t(f) \right) + \int_{a+b-x}^b |t - b| d \left(\bigvee_a^t(f) \right) \\
& = \int_a^x (t - a) d \left(\bigvee_a^t(f) \right) - \int_x^{\frac{a+b}{2}} \left(t - \frac{a+b}{2} \right) d \left(\bigvee_a^t(f) \right) \\
& \quad + \int_{\frac{a+b}{2}}^{a+b-x} \left(t - \frac{a+b}{2} \right) d \left(\bigvee_a^t(f) \right) + \int_{a+b-x}^b (b - t) d \left(\bigvee_a^t(f) \right) \\
& = (x - a) \bigvee_a^x(f) - \int_a^x \left(\bigvee_a^t(f) \right) dt + \left(x - \frac{a+b}{2} \right) \bigvee_a^x(f) + \int_x^{\frac{a+b}{2}} \left(\bigvee_a^t(f) \right) dt \\
& \quad + \left(\frac{a+b}{2} - x \right) \bigvee_a^{a+b-x}(f) - \int_{\frac{a+b}{2}}^{a+b-x} \left(\bigvee_a^t(f) \right) dt - (x - a) \bigvee_a^x(f) + \int_{a+b-x}^b \left(\bigvee_a^t(f) \right) dt \\
& = 2 \left(\frac{3a+b}{4} - x \right) \bigvee_x^{a+b-x}(f) - \int_a^{\frac{a+b}{2}} \operatorname{sgn}(x - t) \bigvee_a^t(f) dt - \int_{\frac{a+b}{2}}^b \operatorname{sgn}(a + b - x - t) \bigvee_a^t(f) dt \\
& = 2 \left(\frac{3a+b}{4} - x \right) \bigvee_x^{a+b-x}(f) - \int_a^{\frac{a+b}{2}} \operatorname{sgn}(x - t) \bigvee_a^t(f) dt + \int_a^{\frac{a+b}{2}} \operatorname{sgn}(x - s) \bigvee_a^{a+b-s}(f) ds \\
& = 2 \left(\frac{3a+b}{4} - x \right) \bigvee_x^{a+b-x}(f) + \int_a^{\frac{a+b}{2}} \operatorname{sgn}(x - t) \bigvee_t^{a+b-t}(f) dt \\
& = (b - a)Q(x),
\end{aligned}$$

for all $x \in [a, \frac{a+b}{2}]$, and the inequality (2.2) is proved.

Now, since \bigvee_a^{\cdot} is monotonic nondecreasing on $[a, b]$, then

$$\begin{aligned}
& \int_a^x \left(\bigvee_a^t(f) \right) dt \geq 0, \quad \int_x^{\frac{a+b}{2}} \left(\bigvee_a^t(f) \right) dt \leq \left(\frac{a+b}{2} - x \right) \bigvee_a^{\frac{a+b}{2}}(f), \\
& \int_{\frac{a+b}{2}}^{a+b-x} \left(\bigvee_a^t(f) \right) dt \geq \left(\frac{a+b}{2} - x \right) \bigvee_a^{\frac{a+b}{2}}(f) \quad \text{and} \quad \int_{a+b-x}^b \left(\bigvee_a^t(f) \right) dt \leq (x - a) \bigvee_a^b(f),
\end{aligned}$$

for all $x \in [a, \frac{a+b}{2}]$, which gives

$$\begin{aligned}
(b - a)Q(x) & \leq 2 \left(\frac{3a+b}{4} - x \right) \bigvee_x^{a+b-x}(f) + (x - a) \bigvee_a^b(f) \\
& = (x - a) \bigvee_a^x(f) + \left(\frac{a+b}{2} - x \right) \bigvee_x^{a+b-x}(f) + (x - a) \bigvee_{a+b-x}^b(f)
\end{aligned}$$

and the inequality (2.3) is proved.

Utilising the second part of the second inequality in (2.1) and Hölder's inequality, we deduce that

$$\begin{aligned}
& (b-a)Q(x) \\
& \leq \left[\bigvee_a^x(f) \right]^{\frac{1}{q}} \left[\int_a^x |t-a|^p d \left(\bigvee_a^t(f) \right) \right]^{\frac{1}{p}} + \left[\bigvee_x^{a+b-x}(f) \right]^{\frac{1}{q}} \left[\int_x^{a+b-x} \left| t - \frac{a+b}{2} \right|^p d \left(\bigvee_a^t(f) \right) \right]^{\frac{1}{p}} \\
& \quad + \left[\bigvee_{a+b-x}^b(f) \right]^{\frac{1}{q}} \left[\int_{a+b-x}^b |t-b|^p d \left(\bigvee_a^t(f) \right) \right]^{\frac{1}{p}} \\
& \leq \left[\bigvee_a^b(f) \right]^{\frac{1}{q}} \left[\int_a^x |t-a|^p d \left(\bigvee_a^t(f) \right) + \int_x^{a+b-x} \left| t - \frac{a+b}{2} \right|^p d \left(\bigvee_a^t(f) \right) \right. \\
(2.8) \quad & \left. + \int_{a+b-x}^b |t-b|^p d \left(\bigvee_a^t(f) \right) \right]^{\frac{1}{p}}.
\end{aligned}$$

Now, observe that

$$\begin{aligned}
R(x) &:= \int_a^x |t-a|^p d \left(\bigvee_a^t(f) \right) + \int_x^{a+b-x} \left| t - \frac{a+b}{2} \right|^p d \left(\bigvee_a^t(f) \right) + \int_{a+b-x}^b |t-b|^p d \left(\bigvee_a^t(f) \right) \\
&= \int_a^x (t-a)^p d \left(\bigvee_a^t(f) \right) + \int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right)^p d \left(\bigvee_a^t(f) \right) \\
& \quad + \int_{\frac{a+b}{2}}^{a+b-x} \left(t - \frac{a+b}{2} \right)^p d \left(\bigvee_a^t(f) \right) + \int_{a+b-x}^b (b-t)^p d \left(\bigvee_a^t(f) \right) \\
&= (x-a)^p \bigvee_a^x(f) - p \int_a^x (t-a)^{p-1} \bigvee_a^t(f) dt - \left(\frac{a+b}{2} - x \right)^p \bigvee_a^x(f) \\
& \quad + p \int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right)^{p-1} \bigvee_a^t(f) dt + \left(\frac{a+b}{2} - x \right)^p \bigvee_a^{a+b-x}(f) \\
& \quad - p \int_{\frac{a+b}{2}}^{a+b-x} \left(t - \frac{a+b}{2} \right)^{p-1} \bigvee_a^t(f) dt - (x-a)^p \bigvee_a^{a+b-x}(f) + p \int_{a+b-x}^b (b-t)^{p-1} \bigvee_a^t(f) dt \\
&= \left[\left(\frac{a+b}{2} - x \right)^p - (x-a)^p \right] \bigvee_x^{a+b-x}(f) \\
& \quad - p \int_a^x (t-a)^{p-1} \bigvee_a^t(f) dt + p \int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right)^{p-1} \bigvee_a^t(f) dt \\
& \quad - p \int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - s \right)^{p-1} \bigvee_a^{a+b-s}(f) ds + p \int_a^x (s-a)^{p-1} \bigvee_a^{a+b-s}(f) ds \\
&= \left[\left(\frac{a+b}{2} - x \right)^p - (x-a)^p \right] \bigvee_x^{a+b-x}(f) - p \int_a^{\frac{a+b}{2}} r_p(x, t) \operatorname{sgn}(x-t) \bigvee_a^t(f) dt \\
& \quad + p \int_a^{\frac{a+b}{2}} r_p(x, s) \operatorname{sgn}(x-s) \bigvee_a^{a+b-s}(f) ds
\end{aligned}$$

$$= \left[\left(\frac{a+b}{2} - x \right)^p - (x-a)^p \right] \bigvee_x^{a+b-x} (f) + p \int_a^{\frac{a+b}{2}} r_p(x, t) \operatorname{sgn}(x-t) \bigvee_t^{a+b-t} (f) dt,$$

where r_p is given in (2.5). Utilising (2.8), we deduce the first part of (2.4).

Since \bigvee_a is monotonic nondecreasing on $[a, b]$, we have

$$\begin{aligned} R(x) &\leq \left[\left(\frac{a+b}{2} - x \right)^p - (x-a)^p \right] \bigvee_x^{a+b-x} (f) + p \left[\int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right)^{p-1} dt \right] \bigvee_a^{\frac{a+b}{2}} (f) \\ &\quad - p \left[\int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - s \right)^{p-1} ds \right] \bigvee_a^{\frac{a+b}{2}} (f) + p \left[\int_a^x (s-a)^{p-1} ds \right] \bigvee_a^b (f) \\ &= \left[\left(\frac{a+b}{2} - x \right)^p - (x-a)^p \right] \bigvee_x^{a+b-x} (f) + (x-a)^p \bigvee_a^b (f) \\ &= (x-a)^p \bigvee_a^x (f) + \left(\frac{a+b}{2} - x \right)^p \bigvee_x^{a+b-x} (f) + (x-a)^p \bigvee_{a+b-x}^b (f), \end{aligned}$$

which proves (2.4). \square

Corollary 2.1. *Under the assumptions of Theorem 2.1 with $x = \frac{3a+b}{4}$, we have the refined trapezoid type inequalities*

$$\begin{aligned} &\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ &\leq \frac{1}{b-a} \int_a^{\frac{a+b}{2}} \operatorname{sgn}\left(\frac{3a+b}{4} - t\right) \bigvee_t^{a+b-t} (f) dt \\ &\leq \frac{1}{b-a} \left[\bigvee_a^b (f) \right]^{\frac{1}{q}} \left[p \int_a^{\frac{a+b}{2}} r_p(x, t) \operatorname{sgn}\left(\frac{3a+b}{4} - t\right) \bigvee_t^{a+b-t} (f) dt \right]^{\frac{1}{p}} \\ (2.9) \quad &\leq \frac{1}{4} \bigvee_a^b (f), \end{aligned}$$

where r_p is given in (2.5).

Corollary 2.2. *Under the assumptions of Theorem 2.1 with $x = a$, we have the refined trapezoid inequalities*

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ &\leq \frac{1}{2} \bigvee_a^b (f) - \frac{1}{b-a} \int_a^{\frac{a+b}{2}} \left(\bigvee_t^{a+b-t} (f) \right) dt \\ &\leq \frac{1}{b-a} \left[\bigvee_a^b (f) \right]^{\frac{1}{q}} \left[\left(\frac{b-a}{2} \right)^p \bigvee_a^b (f) - p \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right)^{p-1} \bigvee_t^{a+b-t} (f) dt \right]^{\frac{1}{p}} \\ (2.10) \quad &\leq \frac{1}{2} \bigvee_a^b (f). \end{aligned}$$

Corollary 2.3. *Under the assumptions of Theorem 2.1 with $x = \frac{a+b}{2}$, we have the refined midpoint inequalities*

$$\begin{aligned}
 & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
 & \leq \frac{1}{b-a} \int_a^{\frac{a+b}{2}} \left(\bigvee_t^{a+b-t}(f) \right) dt \\
 & \leq \frac{1}{b-a} \left[\bigvee_a^b(f) \right]^{\frac{1}{q}} \left[p \int_a^{\frac{a+b}{2}} (t-a)^{p-1} \bigvee_t^{a+b-t}(f) dt \right]^{\frac{1}{p}} \\
 & \leq \frac{1}{2} \bigvee_a^b(f).
 \end{aligned}
 \tag{2.11}$$

Remark 1. *The inequalities (2.2)-(2.4) provide a refinement of the companion of Ostrowski inequality (1.3), and the inequalities (2.10), (2.11) and (2.9) are refinements of the trapezoid inequality, the mid-point inequality and the trapezoid type inequality (1.4), respectively, which were obtained in [8], [9] and [11], respectively.*

A new inequality of Ostrowski's type may be stated as follows:

Corollary 2.4. *Let f be as in Theorem 2.1. Additionally, if f is symmetric about the line $x = \frac{a+b}{2}$, i.e., $f(a+b-x) = f(x)$, then for all $x \in [a, \frac{a+b}{2}]$ we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq Q(x),
 \tag{2.12}$$

where $Q(x)$ satisfies (2.3) and (2.4).

Remark 2. *Under the assumptions of Corollary 2.4 with $x = a$, we have*

$$\begin{aligned}
 & \left| f(a) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
 & \leq \frac{1}{2} \bigvee_a^b(f) - \frac{1}{b-a} \int_a^{\frac{a+b}{2}} \left(\bigvee_t^{a+b-t}(f) \right) dt \\
 & \leq \frac{1}{b-a} \left[\bigvee_a^b(f) \right]^{\frac{1}{q}} \left[\left(\frac{b-a}{2} \right)^p \bigvee_a^b(f) - p \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right)^{p-1} \bigvee_t^{a+b-t}(f) dt \right]^{\frac{1}{p}} \\
 & \leq \frac{1}{2} \bigvee_a^b(f).
 \end{aligned}
 \tag{2.13}$$

3. APPLICATION TO PROBABILITY DENSITY FUNCTIONS

Now, let X be a random variable taking values in the finite interval $[a, b]$, with the probability density function $f : [a, b] \rightarrow [0, 1]$ and with the cumulative distribution function

$$F(x) = \Pr(X \leq x) = \int_a^x f(t) dt.$$

The following results hold:

Theorem 3.1. *With the assumptions of Theorem 2.1, we have*

$$(3.1) \quad \left| \frac{1}{2} [F(x) + F(a+b-x)] - \frac{b-E(X)}{b-a} \right| \leq T(x)$$

for all $x \in [a, \frac{a+b}{2}]$, where $E(X)$ is the expectation of X and

$$(3.2) \quad \begin{aligned} T(x) &:= \frac{1}{b-a} \left[2 \left(\frac{3a+b}{4} - x \right) [F(a+b-x) - F(x)] + \int_a^{\frac{a+b}{2}} \operatorname{sgn}(x-t) [F(a+b-t) - F(t)] dt \right] \\ &\leq \frac{1}{b-a} \left[2 \left(\frac{3a+b}{4} - x \right) [F(a+b-x) - F(x)] + (x-a) \right] \\ &\leq \frac{1}{4} + \left| \frac{x - \frac{3a+b}{4}}{b-a} \right|. \end{aligned}$$

We also have

$$(3.3) \quad \begin{aligned} T(x) &\leq \frac{1}{b-a} \left\{ \left[\left(\frac{a+b}{2} - x \right)^p - (x-a)^p \right] [F(a+b-x) - F(x)] \right. \\ &\quad \left. + p \int_a^{\frac{a+b}{2}} r_p(x,t) \operatorname{sgn}(x-t) [F(a+b-t) - F(t)] dt \right\}^{\frac{1}{p}} \\ &\leq \frac{1}{b-a} \left\{ \left[\left(\frac{a+b}{2} - x \right)^p - (x-a)^p \right] [F(a+b-x) - F(x)] + (x-a)^p \right\}^{\frac{1}{p}} \\ &\leq \frac{1}{4} + \left| \frac{x - \frac{3a+b}{4}}{b-a} \right|, \end{aligned}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and r_p is given in (2.5).

Proof. By (2.2)-(2.4) on choosing $f = F$ and taking into account

$$E(X) = \int_a^b t dF(t) = b - \int_a^b F(t) dt,$$

we obtain (3.1)-(3.3). □

Corollary 3.1. *Under the assumptions of Theorem 3.1 with $x = \frac{3a+b}{4}$, we have*

$$(3.4) \quad \begin{aligned} &\left| \frac{1}{2} \left[F\left(\frac{3a+b}{4}\right) + F\left(\frac{a+3b}{4}\right) \right] - \frac{b-E(X)}{b-a} \right| \\ &\leq \frac{1}{b-a} \int_a^{\frac{a+b}{2}} \operatorname{sgn}\left(\frac{3a+b}{4} - t\right) [F(a+b-t) - F(t)] dt \\ &\leq \frac{1}{b-a} \left[p \int_a^{\frac{a+b}{2}} r_p(x,t) \operatorname{sgn}\left(\frac{3a+b}{4} - t\right) [F(a+b-t) - F(t)] dt \right]^{\frac{1}{p}} \\ &\leq \frac{1}{4}, \end{aligned}$$

where r_p is given in (2.5).

Remark 3. *The inequalities (3.1)-(3.4) provide refinements of the inequalities given in [11, Theorem 5] and [11, Corollary 4].*

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